ON THE HILBERT 17TH PROBLEM FOR GLOBAL ANALYTIC FUNCTIONS

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January 26, 2004

Dedicated to Eberhard Becker
on the occasion of his 60th birthday

Abstract

We consider Hilbert's 17 problem for global analytic functions in a modified form that involves infinite sums of squares. This reveals an essential connection between the solution of the problem and the computation of Pythagoras numbers of meromorphic functions.

AMS Subject Classification: Primary 14P99; Secondary 11E25, 32B10.
Keywords: Hilbert 17th Problem, Pythagoras number, sum of squares, bad set, germs at closed sets.

1 Introduction

Of all possible versions of the famous Hilbert 17th Problem, that for global analytic functions is the one standing apart from any substantial progress. As is well known, the problem is whether:

H17. Every positive semidefinite analytic function $f : \mathbb{R}^n \to \mathbb{R}$ is a finite sum of squares.

In fact, the best result we can state today goes back to the early '80s: *a positive semidefinite global analytic function whose zero set is discrete off a compact set is a sum of squares of meromorphic functions,* ([BKS] and [Rz],[Jw], see also [ABR]). Of course, the sum of squares here is a *finite sum.* However, as we are dealing with analytic functions, also infinite sums have a meaning, as one can consider *convergent series of squares.* Quite obviously, we must be very careful with the meaning of convergent here, but this we postpone a little. Clearly, infinite sum of squares, whenever defined, are to be positive semidefinite, and we can conversely ask whether:

h17. Every positive semidefinite analytic function $f : \mathbb{R}^n \to \mathbb{R}$ is an infinite sum of squares.

*All authors supported by European RAAG HPRN-CT-2001-0027; first and second named authors also by Italian GNSAGA of INdAM and MIUR, third and fourth by Spanish GAAR BFM-2002-04797.

This is a weaker version of Hilbert's Problem, but all the same a qualitative question. In addition, we can consider the quantitative aspect of *finding the smallest number* $p_{\mathbb{R}^n} \leq +\infty$ *of squares needed to represent any sum of squares.* This $p_{\mathbb{R}^n}$ is called *Pythagoras number,* and might indeed be infinite either because there are finite sums of squares of arbitrary length, or just because there is some infinite sum of squares which is not finite. The first case is not new, but the second arises from the consideration of convergent series of squares, and we separate the following property:

Finiteness. Every infinite sum of squares is also a finite sum of squares.

With this terminology, H17 is equivalent to h17 plus finiteness, and quite remarkably, we will see that *finiteness is equivalent to* $p_{\mathbb{R}^n} < +\infty$. In other words, if infinite sums of squares are all finite, then finite sums of squares are all sums of p squares for some fixed integer p. In particular, this means that the solution to Hilbert's 17th Problem H17 is in the affirmative only if the Pythagoras number is finite. This, and other quantitative implications, are consequences of our main theorem, which localizes the obstruction for a positive semidefinite function f to be an infinite sum of squares: roughly, the obstruction concentrates on the connected components of the zero set $\{f = 0\}$. The suitable framework for precise formulations is provided by germs at closed sets. To this we now turn, in the more general context of arbitrary manifolds.

In what follows, we consider a real analytic manifold $M \subset \mathbb{R}^n$ (which we can suppose embedded as a closed set). This embedding dimension n will appear in various bounds in our results; the dimension of M will be denoted by m.

(1.1) Germs at a closed set $Z \subset M$**.** Germs at Z are defined exactly as germs at a point, through neighborhoods of Z in M; we will denote by f_Z the germ at Z of an analytic function f defined in some neighborhood of Z. We have the ring $\mathcal{O}(M_Z)$ of analytic function germs at Z, and its total ring of fractions $\mathcal{M}(M_Z)$, which is the ring of meromorphic function germs at Z. Note that for $Z = M$ we get nothing but global analytic and global meromorphic functions on M. If Z is connected, then $\mathcal{O}(M_Z)$ is a domain and $\mathcal{M}(M_Z)$ a field.

As usual, a germ f_Z is *positive semidefinite* when some representative f is positive semidefinite on some neighborhood of Z.

Next, we must define properly infinite sums of squares. The first attempt to use convergent, even uniformly convergent, series of squares cannot work, as in the real case uniform convergence does not guarantee analyticity. As we really need to operate freely with these infinite sums, we must resort to complexification, which on the other hand is costumary in real analytic geometry. Thus, we introduce the following definition:

Definition 1.2 *Let* $Z \subset M$ *be closed. An* infinite sum of squares of analytic function germs at Z *is a series* $\sum_{k\geq 1} f_k^2$ *where all* $f_k \in \mathcal{O}(M_Z)$*, such that:*

(i) The f_k*'s have holomorphic extensions* F_k*'s, all defined in the same neighborhood* V *of* Z *in some complexification of* M*, and*

(ii) For every compact set $L \subset V$, $\sum_{k\geq 1} \sup_L |F_k|^2 < +\infty$.

The condition *(ii)* is the standard bound one uses to check that a function series is absolutely and uniformly convergent on compact sets. Accordingly, the infinite sum $\sum_{k\geq 1} f_k^2$ defines well an analytic function f on $\Omega = V \cap M$, which is a neighborhood of Z, and hence we have an analytic function germ f_Z: we write $f_Z = \sum_{k\geq 1} f_k^2 \in \mathcal{O}(M_Z)$. Hence, it makes sense to say that an element of the ring $\mathcal{O}(M_Z)$ is a sum of p squares in $\mathcal{O}(M_Z)$, even for $p = +\infty$. (Of course, this discussion includes finite sums of squares.)

Next, we consider meromorphic functions:

Definition 1.3 *Let $Z \subset M$ be closed. An analytic function germ f_Z is a* sum of $p \leq +\infty$ squares of meromorphic function germs at Z *if there is $g_Z \in \mathcal{O}(M_Z)$ such that $g_Z^2 f_Z$ is a sum of p squares of analytic function germs at Z. The zero set $\{g_Z = 0\}$ is called the* bad set *of the sum of squares.*

The above notion of bad set mimics the terminology introduced in [Dz], but notice that here we refer to each given sum of squares, not to the function it represents.

The fact that a germ f_Z may have different representations as a sum of squares, either finite or infinite, is a part of Hilbert's 17 Problem. The choice of a suitable sum of squares representation is always a crucial matter, often made to have a *controlled bad set,* that is, to have a bad set contained in the zero set germ $\{f_Z = 0\}$.

The central result in this paper is the following:

Theorem 1.4 *Let $f : M \to \mathbb{R}$ be a positive semidefinite analytic function, and let $q \leq +\infty$. Suppose that for every connected component Y of the zero set $\{f = 0\}$, the germ f_Y is a sum of q squares with controlled bad set. Then f is a sum of $q+1$ squares with controlled bad set.*

Concerning the difference between arbitrary and controlled bad sets we will prove:

Proposition 1.5 *Let $Z \subset M$ be closed, and f_Z an analytic function germ which is a sum of p squares of meromorphic function germs. Then f_Z is a sum of $2^n p$ squares with controlled bad set. The number of squares can be lowered to $2^{n-1}p$ if f_Z vanishes on Z.*

From this and 1.4 we deduce:

Corollary 1.6 *Let $f : M \to \mathbb{R}$ be a positive semidefinite analytic function. Suppose that for every connected component Y of the zero set $\{f = 0\}$, the germ f_Y is a sum of $p \leq +\infty$ squares. Then, f is a sum of $2^{n-1}p + 1$ squares with controlled bad set.*

Note the following immediate but relevant consequence: *to represent f as a sum of squares it suffices to represent its restriction to some neighborhood of its zero set.*

The only general result we know so far is that if Y is compact the positive semidefinite germ f_Y is a finite sum of squares [ABR], hence we deduce:

Corollary 1.7 *Let $f : M \to \mathbb{R}$ be a positive semidefinite analytic function, such that all connected components of its zero set $\{f = 0\}$ are compact. Then f is a sum of squares.*

But notice that the sum here might well be infinite, since we have no bound for the number of squares needed to represent each germ f_Y. In one case we do know such a bound: when Y is a singleton, f_Y is a sum of $p = 2^m + m$ squares ($m = \dim M$, because a suitable modification of the germ is algebraic, see [BKS])). In view of this, the result stated at the very beginning follows readily from 1.4:

Corollary 1.8 *Let $f : M \to \mathbb{R}$ be a positive semidefinite analytic function, such that the set $\{f = 0, \|x\| \geq \rho\}$ is discrete for some $\rho > 0$. Then f is a finite sum of squares.*

Now we formulate in a more technical way the qualitative and quantitative Hilbert problems:

> **H17 for** $Z \subset M$**.** Every positive semidefinite analytic function germ at Z is a finite sum of squares of meromorphic function germs at Z.
>
> **h17 for** $Z \subset M$**.** Every positive semidefinite analytic function germ at Z is a (possibly infinite) sum of squares of meromorphic function germs at Z.

This is a relative notion that refers to germs, and it must be clear where Z is contained, for instance. Only in case $Z = M$, we write simply *for* M. With this terminology, from Theorem 1.4 and Proposition 1.5 we deduce:

Corollary 1.9 *If* h17 *holds for all proper global analytic sets $Y \subset M$, then it holds for M.*

In fact, by factoring out codimension 1 components *it is enough to consider global analytic sets $Y \subset M$ of codimension ≥ 2.*

As a kind of converse we will prove:

Proposition 1.10 *Suppose* h17 *holds for $M \times \mathbb{R}$, then it holds for*
(a) all closed sets $Z \subset M$, and
(b) all closed sets $Z \subset \mathbb{R}^m$, $m = \dim(M)$.

The proofs of these statements use suitable closed embeddings into $M \times \mathbb{R}$, and need *the control of bad sets to restrict sums of squares without spoiling denominators.* One can say more for $M = \mathbb{R}^3$:

Proposition 1.11 *If* h17 *holds for $\mathbb{R}^3$, then it holds for all closed sets $Z \subset \mathbb{R}^3$.*

Back to the above formulations, H17 implies h17 trivially, and, as every sum of squares is positive semidefinite, H17 also implies this:

Finiteness at Z. Every infinite sum of squares of meromorphic function germs at Z is also a finite sum of squares of meromorphic function germs at Z.

Conversely, h17 and this finiteness property together imply H17. Consequently, let us look closer to finiteness, to decipher its quantitative content keeping in mind that all that quantitative content is actually a content of H17. We stress how this differs from the algebraic case, where Pythagoras numbers stand quite apart from the qualitative question [BCR, 6].

Firstly, denote by p_Z the smallest number $p \leq +\infty$ such that every sum of squares of meromorphic function germs at Z is a sum of p squares. This number p_Z will be called Pythagoras number, but notice that it is not exactly the Pythagoras number $p(M_Z)$ of the ring $\mathcal{M}(M_Z)$. Indeed, the definition of the latter refers to *finite* sums of squares and consequently we can only write $p_Z \geq p(M_Z)$. Of course, both Pythagoras numbers coincide if H17 holds at Z, but they might be different in case finiteness fails and $p(M_Z) < +\infty$. Whence, the question is whether or not $p_Z < +\infty$, and how to get a sharp bound, which seems very difficult for the moment. For instance, although H17 holds for compact sets, we do not know whether at them the Pythagoras number is finite. But from Corollary 1.6 we deduce:

Corollary 1.12 *If there is an integer p such that $p_Y \leq p$ for all proper global analytic sets $Y \subset M$, then $p_M \leq 2^{n-1}p + 1$. In particular, if $p < +\infty$, finiteness holds for M and $p(M) \leq 2^{n-1}p + 1$.*

Again, codimension 1 components are not that relevant, and we can rely only on global analytic sets $Y \subset M$ of codimension ≥ 2. However, this increases the bound $2^{n-1}p + 1$ to $r(2^{n-1}p + 1)$, where r stands for *the minimum number of global generators of all locally principal analytic sheaves on M*.

As a matter of fact, whether or not the Pythagoras numbers are finite is one main content of H17, despite H17 does not seem at first sight a quantitative matter. Indeed, we will prove the following:

Proposition 1.13 *Finiteness for $M \times \mathbb{R}$ implies $p(M_Z) = p_Z < +\infty$ (in particular finiteness) for all closed sets $Z \subset M$.*

This is proved by means of suitable closed embeddings into $M \times \mathbb{R}$, and uses the fact that *finiteness is inherited by closed submanifolds*. The above result has two particular cases of interest:

1. For $Z = M$, $p(M_Z)$ is the Pythagoras number of the field of global meromorphic functions on M.
2. For a singleton $Z = \{a\}$, $p(M_Z)$ is the Pythagoras number of the field of meromorphic function germs at the point a.

Note here that the latter field is the field $\mathcal{M}_m = \mathbb{R}(\{x_1, \ldots, x_m\})$ of meromorphic power series in $m = \dim(M)$ variables, and the computation of the Pythagoras number $p(\mathcal{M}_m)$ of this field is an important old open problem in the theory of quadratic forms: the only bound known (for $m \geq 3$) is $p_3 \leq 8$, and even finiteness remains open for larger m. To this respect, in our vein here we see from 1.13 that H17 for $\mathbb{S}^m \times \mathbb{R}$ implies $p(\mathcal{M}_m) < +\infty$.

A variation of Proposition 1.13 is this:

Proposition 1.14 *Suppose that $p(M \times \mathbb{R}) < +\infty$. Then $p(M_Z) < +\infty$ with a common bound for all closed sets $Z \subset M$.*

We stress that the different hypotheses in Propositions 1.13 and 1.14 correspond to the existence of a common bound for the Pythagoras numbers. To have all in a single condition, we need finiteness for $M \times \mathbb{R}^2$, not merely for $M \times \mathbb{R}$. However, we obtain something better for $\mathbb{R}^m$, $m = \dim(M)$:

Proposition 1.15 *Finiteness for $M \times \mathbb{R}$ implies $p(\mathbb{R}^m_Z) = p_Z < +\infty$ (in particular finiteness) with a common bound for all closed sets $Z \subset \mathbb{R}^m$.*

These results are not best for $M = \mathbb{R}^m$. Indeed, the main matter here is to distribute sparsely closed sets, and this can be done very well in $\mathbb{R}^m$. Thus, we will improve on the two particular cases of Proposition 1.13 and show this:

Proposition 1.16 *Suppose finiteness for $\mathbb{R}^m$. Then*

$$p(\mathbb{R}^m) < +\infty \qquad \textit{and} \qquad p(\mathcal{M}_m) < +\infty.$$

Thus, we do not need to pass from $M = \mathbb{R}^m$ to $M \times \mathbb{R} = \mathbb{R}^{m+1}$, as in Proposition 1.13. For instance, we knew that $p(\mathcal{M}_m) < +\infty$ followed from H17 for $\mathbb{R}^{m+1}$, but now we see that it follows from H17 for $\mathbb{R}^m$. Another example of this improvement is that we get a common bound for all p_Z's from finiteness for $\mathbb{R}^{m+1}$ instead of $\mathbb{R}^{m+2}$.

And for $M = \mathbb{R}^3$ we can add this:

Proposition 1.17 *Finiteness for $\mathbb{R}^3$ implies $p(\mathbb{R}^3_Z) = p_Z < +\infty$ (in particular finiteness) with a common bound for all closed sets $Z \subset \mathbb{R}^3$.*

Summing up all we know for $\mathbb{R}^3$, we get:

Proposition 1.18 *The following assertions are equivalent:*

(i) H17 *holds for* $\mathbb{R}^3$

(ii) H17 *holds for all closed sets $Z \subset \mathbb{R}^3$ and $p(\mathbb{R}^3_Z) < +\infty$ with a common bound for all Z's.*

(iii) H17 *holds for all closed analytic curves* $X \subset \mathbb{R}^3$ *and* $p(\mathbb{R}^3_X) < +\infty$ *with a common bound for all* X*'s.*

The paper is organized as follows. In Section 2 we prove some crucial lemmas concerning holomorphic functions: Lemma 2.2 will be used to separate the connected components of a given zero set, Lemma 2.3 describes how to extend holomorphic functions with fixed values on a given zero set, and Lemma 2.4 does the same with sums of squares. Section 3 is devoted to Theorem 3.1, which contains the bulk of the glueing and globalization techniques behind the scenes. From this, the main result Theorem 1.4 is deduced in Section 4, which also includes some additional improvements concerning the simplification of codim 1 factors. In Section 5 we prove Proposition 1.5, and deduce from it Proposition 1.10. Next, Section 6 is devoted to the finiteness implications of Hilbert's 17th Problem. The special results for $M = \mathbb{R}^3$ (Propositions 1.11 and 1.17) are proved in Section 7, after a discussion of positive semidefinite germs with small zero set.

The authors would like to thank Prof. M. Shiota for friendly helpful discussions during the preparation of this work, notably in connection with the important fact that bad sets can be controlled.

2 Preliminaries on holomorphic functions

We gather here some notations and technical lemmas for later purposes. Although our problem concerns real analytic functions, we will of course use some complex analysis. For holomorphic functions we refer the reader to the classical [GuRo].

(2.1) General terminology. In what follows we denote the coordinates in $\mathbb{C}^n$ by $z = (z_1, \ldots, z_n)$, with $z_i = x_i + \sqrt{-1}y_i$, where $x_i = \mathrm{Re}(z_i)$ and $y_i = \mathrm{Im}(z_i)$ are respectively the *real* and the *imaginary parts* of z_i. Also, we consider the usual conjugation $\sigma : \mathbb{C}^n \to \mathbb{C}^n, z \mapsto \overline{z} = (\overline{z_1}, \ldots, \overline{z_n})$, whose fixed points are $\mathbb{R}^n$. We say that a subset $A \subset \mathbb{C}^n$ is *(σ-)invariant* if $\sigma(Y) = Y$; clearly, $A \cap \sigma(A)$ is the biggest invariant subset of A. Thus, we see real spaces as subsets of complex spaces. We will use the notations Int and Cl to denote topological interiors and closures, respectively. Given a positive real number $\rho > 0$ we denote

$$\Delta_\rho = \{z \in \mathbb{C}^n : |z_1| < \rho, \ldots, |z_n| < \rho\}, \quad \mathrm{Cl}_{\mathbb{C}^n}(\Delta_\rho) = \{z \in \mathbb{C}^n : |z_1| \leq \rho, \ldots, |z_n| \leq \rho\}.$$

Let $U \subset \mathbb{C}^n$ be an invariant open set and let $F : U \to \mathbb{C}$ be a holomorphic function. We say that F is *(σ-)invariant* if $F(z) = \overline{F(\overline{z})}$. This implies that F restricts to a real analytic function on $U \cap \mathbb{R}^n$. In general, we denote by:

$$\begin{array}{rccc} \Re(F): & U & \to & \mathbb{C} \\ & z & \mapsto & \frac{F(z)+\overline{F(\overline{z})}}{2} \end{array} \qquad \begin{array}{rccc} \Im(F): & U & \to & \mathbb{C} \\ & z & \mapsto & \frac{F(z)-\overline{F(\overline{z})}}{2\sqrt{-1}} \end{array}$$

the *real* and the *imaginary* parts of F, which satisfy $F = \Re(F) + \sqrt{-1}\,\Im(F)$. Note that both are invariant holomorphic functions. $\square$

In order to analize the connected components of a zero set, we will consider suitable Stein neighbourhoods $\mathcal{U}$ of $\mathbb{R}^n$ in $\mathbb{C}^n$:

Lemma 2.2 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a analytic function. Then, there exists a basis of open invariant Stein neighbourdhoods $\mathcal{U}$ of $\mathbb{R}^n$ in $\mathbb{C}^n$ such that:*

a) $\mathbb{R}^n$ is a deformation retract of $\mathcal{U}$,

b) f has an holomorphic extension F defined on $\mathcal{U}$,

c) For each connected component S of $F^{-1}(0)$ with $S \cap \mathbb{R}^n \neq \emptyset$, this intersection is a connected component Y of $f^{-1}(0)$, and

d) For each compact set $K \subset \mathcal{U}$ such that $K \cap S = \emptyset$, there exists an invariant holomorphic function $\Lambda : \mathcal{U} \to \mathbb{C}$ such that

- *$\Lambda^{-1}(0) = S$, and S is the connected component of $(F + \Lambda^2)^{-1}(0)$ that contains Y,*
- *there is a holomorphic unit v defined on an open neighborhood $V \subset \mathcal{U}$ of S such that $F + \Lambda^2 = vF$.*
- *$F + \Lambda^2$ has no zero in K, and*
- *$\lambda = \Lambda|_{\mathbb{R}^n}$ is a positive semidefinite real analytic function.*

Proof. Let $\{Y_k\}_k$ be the connected components of $f^{-1}(0)$. Take an open neighborhood U of $\mathbb{R}^n$ in $\mathbb{C}^n$ to which f extends holomorphically, denote by F such an extension, and choose for each k an open neighborhood U_k of Y_k in U such that $U_k \cap U_\ell = \emptyset$ if $\ell \neq k$.

Consider the open set $U' = (U \setminus F^{-1}(0)) \cup \bigcup_k U_k$ in $\mathbb{C}^n$ and take an invariant open Stein neighborhood $\mathcal{U} \subset U'$ of $\mathbb{R}^n$ such that $\mathbb{R}^n$ is a deformation retract of $\mathcal{U}$ (see [Ca]). In this situation conditions *a), b)* and *c)* can be checked straightforwardly. As for *d),* we argue as follows.

Let $\mathcal{J}$ be the subsheaf of the sheaf $\mathcal{O}_{\mathcal{U}}$ of holomorphic function germs on $\mathcal{U}$ defined by

$$\mathcal{J}_x = \begin{cases} F_x \cdot \mathcal{O}_{\mathcal{U},x} & \text{if} \quad x \in S \\ \mathcal{O}_{\mathcal{U},x} & \text{if} \quad x \notin S. \end{cases}$$

The open set $\mathcal{U}$ is a Stein manifold, hence $H^1(\mathcal{U}, \mathcal{O}^*_{\mathbb{C}}) = H^2(\mathcal{U}, \mathbb{Z})$, and this group is trivial because $\mathbb{R}^n$ is a deformation retract of $\mathcal{U}$. Consequently, all locally principal coherent sheaves are in fact globally principal. In particular, $\mathcal{J}$ is generated by a holomorphic function $H : \mathcal{U} \to \mathbb{C}$. Consider then $A = \Re(H)$ and $B = \Im(H)$; note that $Y \subset A^{-1}(0) \cap B^{-1}(0) \subset H^{-1}(0) = S$.

Let $\Lambda = \mu(A^2 + B^2)$ for a certain positive real number $\mu > 0$ that we will choose later. Since $\Lambda(z) = \mu H(z)\,\overline{H \circ \sigma(z)}$ for all $z \in \mathcal{U}$, we have $\Lambda(z) = 0$ if and only if $H(z) = 0$ or $H \circ \sigma(z) = 0$; that is, $z \in S$ or $\overline{z} \in S$. But S is invariant (because F and $\mathcal{U}$ are so), hence $z \in S$. Thus, $\Lambda^{-1}(0) = S$.

Now, by construction, we have $F = \Psi H$ for some holomorphic unit Ψ on an open neighborhood V of S, hence:

$$F + \Lambda^2 = F + \mu^2 H^2\,\overline{H^2 \circ \sigma} = \left(1 + \frac{\mu^2 H\,\overline{H^2 \circ \sigma}}{\Psi}\right) F.$$

Obviously $v = 1 + \frac{\mu^2 H\,\overline{H^2 \circ \sigma}}{\Psi}$ is a well defined holomorphic unit in a neighborhood of S, say V, after shrinking.

Next, we choose μ. Since the zeros of the holomorphic function $A^2 + B^2$ are all in S and $K \cap S = \emptyset$, we can take

$$\mu = +\sqrt{\frac{1 + \max_K |F|}{\min_K |A^2 + B^2|^2}} > 0$$

so that $|F| < \mu^2 |A^2 + B^2|^2$ on K. Hence, $F + \Lambda^2$ has no zero in K.

Let us check that the connected component T of $(F + \Lambda^2)^{-1}(0)$ that contains Y is S. Clearly $Y \subset S \subset T$. Suppose that $S \neq T$, say $a \in T \setminus S$, and pick $b \in S$. Since T is connected there is a path $\gamma : [0,1] \to T$ such that $\gamma(0) = a$ and $\gamma(1) = b$. Let $0 < s = \min\{t \in [0,1] : \gamma(t) \in S\}$. Since $z = \gamma(s) \in S \subset V$, the germs at z of $F + \Lambda^2$ and F differ by a unit, hence the set germs T_z and S_z coincide. But this is impossible because $\gamma[0,s) \subset T \setminus S$.

The last assertion of the statement is clear from the definition of Λ, and we are done.

□

We next see how to extend an holomorphic function modulo another with some control on its behaviour.

Lemma 2.3 *Let $\mathcal{U}$ be an invariant open Stein neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ and let $\Phi : \mathcal{U} \to \mathbb{C}$ be an invariant holomorphic function. Let V be an invariant open neighbourhood of the connected components of $\Phi^{-1}(0)$ that meet $\mathbb{R}^n$, and suppose that V does not meet the other connected components. Let $K \subset \mathcal{U}$ be an invariant compact set. Then there exist a real constant $\mu > 0$ and an invariant compact set $L \subset V$ for which the following property holds:*

(∗) *for every invariant holomorphic function $C : V \to \mathbb{C}$ there exists an invariant holomorphic function $A : \mathcal{U} \to \mathbb{C}$ such that $\Phi|_V$ divides $A|_V - C$ and*

$$\sup_K |A| < \mu \sup_L |C|.$$

Proof. First, consider the coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_{\mathcal{U}}^{\mathbb{C}}$ generated by Φ, and the exact sequence of coherent sheafs

$$0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{U}} \to \mathcal{O}_{\mathcal{U}}/\mathcal{J} \to 0.$$

Now, we have a corresponding diagram of cross sections:

$$\begin{tikzcd}
\mathcal{J}(\mathcal{U}) \arrow[r] \arrow[d] & \mathcal{O}(\mathcal{U}) \arrow[r] \arrow[d] & \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}}/\mathcal{J}) \arrow[d] \\
\mathcal{J}(V) \arrow[r] & \mathcal{O}(V) \arrow[r] & \Gamma(V, \mathcal{O}_{\mathcal{U}}/\mathcal{J})
\end{tikzcd}$$

Here, the upper right arrow is onto because $\mathcal{U}$ is Stein. Furthermore, the right vertical arrow is onto too. Indeed, each cross section of $\mathcal{O}_{\mathcal{U}}/\mathcal{J}$ on V can be extended by zero to $\mathcal{U}$, because the support of $\mathcal{O}_{\mathcal{U}}/\mathcal{J}$ in V is closed in $\mathcal{U}$. Hence we have a linear surjective homomorphism

$$\varphi : \mathcal{O}(\mathcal{U}) \longrightarrow \mathcal{O}(V)/\mathcal{J}(V) \equiv \Gamma(V, \mathcal{O}_{\mathcal{U}}/\mathcal{J}).$$

We equip these vector spaces with their natural topologies. As is well known $\mathcal{O}(\mathcal{U})$ and $\mathcal{O}(V)$ are Frechet spaces with the topology of the uniform convergence on compact sets. Also, by the closure of modules theorem, we know that $\mathcal{J}(V)$ is a closed subspace of $\mathcal{O}(V)$, and $\mathcal{O}(V)/\mathcal{J}(V)$ is also a Frechet space with the quotient topology. Summing up, φ is a continuous surjective homomorphism of Frechet spaces, consequently open [Sc, III.1.2]. In order to make use of this, we describe explicitly the topologies involved.

Let $\{K_i\}_i$ and $\{L_i\}_i$ be families of invariant compact sets in $\mathcal{U}$ and V, such that:

- $\mathrm{Int}_{\mathbb{C}^n}(L_1) \neq \emptyset$,
- $L_i \subset K_i$ for all i,
- $L_i \subset L_{i+1}$ and $K_i \subset K_{i+1}$ for all i, and
- $\bigcup_i \mathrm{Int}_{\mathbb{C}^n}(L_i) = V$ and $\bigcup_i \mathrm{Int}_{\mathbb{C}^n}(K_i) = \mathcal{U}$.

Then the topology of $\mathcal{O}(\mathcal{U})$ (resp. $\mathcal{O}(V)$) is defined by the pseudonorm:

$$\|F\| = \textstyle\sum_i \frac{1}{2^i} \frac{\sup_{K_i}|F|}{1+\sup_{K_i}|F|} \qquad \text{for} \quad F \in \mathcal{O}(\mathcal{U})$$

$$\text{(resp. } \|G\|' = \textstyle\sum_i \frac{1}{2^\ell} \frac{\sup_{L_i}|G|}{1+\sup_{L_i}|G|} \qquad \text{for} \quad G \in \mathcal{O}(V) \text{).}$$

Moreover, by [Sc, I.6.3], the quotient topology of $\mathcal{O}(V)/\mathcal{J}(V)$ is given by the following third pseudonorm:

$$\|\xi\|^* = \inf_G \{\|G\|' : \xi = G + \mathcal{J}(V)\} \quad \text{for} \quad \xi \in \mathcal{O}(V)/\mathcal{J}(V).$$

Next, given the compact set $K \subset \mathcal{U}$, we have the open subset of $\mathcal{O}(\mathcal{U})$ given by

$$W = \{H \in \mathcal{O}(\mathcal{U}) : \sup_K |H| < 1\}.$$

Since φ is open, $\varphi(W)$ is an open neigborhood of 0 in $\mathcal{O}(V)/\mathcal{J}(V)$, and, there exists $\varepsilon > 0$ such that

$$W^* = \{\xi : \|\xi\|^* < \varepsilon\} \subset \varphi(W).$$

Then, we pick $\mu > \frac{2}{\varepsilon}$, and $L = L_i$ with i such that $\sum_{j>i} \frac{1}{2^j} < \frac{\varepsilon}{2}$. We will prove the condition $(*)$ in the statement for such $\mu > 0$ and $L \subset V$, which by construction depend only on K.

Let $C \in \mathcal{O}(V)$ a non-zero holomorphic function. Since the interior of L in $\mathbb{C}^n$ is not empty, $a = \sup_L |C| > 0$, and we denote $G = \frac{1}{a\mu} C \in \mathcal{O}(V)$. Then $\sup_L |G| = \frac{1}{\mu} < \frac{\varepsilon}{2}$, and we have:

$$\begin{aligned}\|G\|' = \sum_j \frac{1}{2^j} \frac{\sup_{L_j} |G|}{1 + \sup_{L_j} |G|} &= \sum_{j=1}^{i} \frac{1}{2^j} \frac{\sup_{L_j} |G|}{1 + \sup_{L_j} |G|} + \sum_{j>i} \frac{1}{2^j} \frac{\sup_{L_j} |G|}{1 + \sup_{L_j} |G|} \\ &< \frac{\sup_L |G|}{1 + \sup_L |G|} \sum_{j=1}^{i} \frac{1}{2^j} + \sum_{j>i} \frac{1}{2^j} < \sup_L |G| \sum_{j=1}^{i} \frac{1}{2^j} + \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

Whence, setting $\xi = G + \mathcal{J}(V)$, we get

$$\|\xi\|^* \leq \|G\|' < \varepsilon,$$

and $\xi \in W^* \subset \varphi(W)$. Consequently, there exists $H \in W$ such that $\varphi(H) = \xi$, that is $H|_V - G \in \mathcal{J}(V)$, and the holomorphic function $F = a\mu H \in \mathcal{O}(\mathcal{U})$ verifies the required conditions. For, $F|_V - C = a\mu(H|_V - G) \in \mathcal{J}(V)$ and since $\sup_K |H| < 1$:

$$\sup_K |F| = a\mu \sup_K |H| < a\mu = \mu \sup_L |C|$$

Finally, if C invariant we take $A = \Re(F)$, and A satisfies the same conditions. First,

$$A|_V - C = \Re(F|_V) - C = \Re(F|_V - C) = \Re(\Lambda \Phi|_V) = \Re(\Lambda)\Phi|_V \in \mathcal{J}(V)$$

for some $\Lambda \in \mathcal{O}(V)$. Secondly, as K is invariant:

$$\sup_K |A| = \sup_K |\Re(F)| = \sup_K \left| \frac{F + \overline{F \circ \sigma}}{2} \right| \leq \sup_K \frac{|F| + |\overline{F \circ \sigma}|}{2} \leq \sup_K |F| < \mu \sup_L |C|,$$

and the proof is complete. $\square$

Now we apply this to infinite sums of squares:

Proposition 2.4 *Let $\mathcal{U}$ be an invariant open Stein neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ and let $\Phi : \mathcal{U} \to \mathbb{C}$ be an invariant holomorphic function. Let V be an open invariant neighbourhood of the connected components of $\Phi^{-1}(0)$ that meet $\mathbb{R}^n$, and suppose that V does not meet the other connected components. Let $C_k : V \to \mathbb{C}$ be a family of invariant holomorphic functions such that $\sum_k \sup_L |C_k|^2 < +\infty$ for every compact set $L \subset V$. Then there exist invariant holomorphic functions $A_k : \mathcal{U} \to \mathbb{C}$, such that $\sum_k \sup_K |A_k|^2 < +\infty$ for every compact set $K \subset \mathcal{U}$ and $\Phi|_V$ divides all the differences $A_k|_V - C_k$.*

Proof. Let $\{K_i\}$ be a family of invariant compact sets such that

- $K_i \subset K_{i+1}$ for all i, and
- $\bigcup_i \mathrm{Int}_{\mathbb{C}^n}(K_i) = \mathcal{U}$.

By 2.3, for each i there exists $\mu_i > 0$ and a compact set $L_i \subset V$ such that if $C \in \mathcal{O}(V)$ is invariant there exists $A \in \mathcal{O}(\mathcal{U})$ invariant such that $A|_V - C = \Phi|_V B$ for some $B \in \mathcal{O}(V)$ and $\sup_{K_i} |A| < \mu_i \sup_{L_i} |C|$. We may assume that $L_i \subset L_{i+1}$ for all i.

Since $\sum_k \sup_{L_i} |C_k|^2 < +\infty$ for all i, there exist a strictly increasing sequence (k_i) of positive integers such that

$$\sum_{k \geq k_i} \sup_{L_i} |C_k|^2 < \frac{1}{2^i \mu_i^2}$$

For each k such that $k_i \leq k < k_{i+1}$ there exists a holomophic function $A_k : \mathcal{U} \to \mathbb{C}$ such that $\sup_{K_i} |A_k| < \mu_i \sup_{L_i} |C_k|$ and $\Phi|_V$ divides $A_k|_V - C_k$. Let us see that for every compact set $K \subset \mathcal{U}$ the series $\sum_k \sup_K |A_k|^2 < +\infty$. Since $\bigcup_i \mathrm{Int}_{\mathbb{C}^n}(K_i) = \mathcal{U}$, it is enough to check that $\sum_k \sup_{K_i} |A_k|^2 < +\infty$ for all i. But,

$$\begin{aligned}
\sum_k \sup_{K_i} |A_k|^2 &= \sum_{1 \leq k < k_i} \sup_{K_i} |A_k|^2 + \sum_{j \geq i} \Big(\sum_{k_j \leq k < k_{j+1}} \sup_{K_i} |A_k|^2 \Big) \\
&\leq \sum_{1 \leq k < k_i} \sup_{K_i} |A_k|^2 + \sum_{j \geq i} \Big(\sum_{k_j \leq k < k_{j+1}} \sup_{K_j} |A_k|^2 \Big) \\
&\leq \sum_{1 \leq k < k_i} \sup_{K_i} |A_k|^2 + \sum_{j \geq i} \Big(\sum_{k \geq k_j} \mu_j^2 \sup_{L_j} |C_k|^2 \Big) \\
&< \sum_{1 \leq k < k_i} \sup_{K_i} |A_k|^2 + \sum_{j \geq i} \mu_j^2 \frac{1}{2^j \mu_j^2} \leq \sum_{1 \leq k < k_{i+1}} \sup_{K_i} |A_k|^2 + 1 < +\infty
\end{aligned}$$

This concludes the proof. □

Remark 2.5 The previous statement can be refined to have a relative version, in the following sense: *if C_1 is divisible on V by some holomorphic function $H : \mathcal{U} \to \mathbb{C}$, the function A_1 can be chosen divisible on $\mathcal{U}$ by H.*

Indeed, notice that the convergence bound does not depend on the choice of a single term of the series. Then, we write $C_1 = C_1^* H$, where C_1^* is holomorphic on V, and by 2.3 there is an holomorphic function A_1^* such that $\Phi|_V$ divides $A_1^*|_V - C_1^*$. Whence, we conclude by taking $A_1 = A_1^* H$. $\square$

3 Glueing and globalization techniques

The purpose of this section is to prove the following key result:

Theorem 3.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positive semidefinite analytic function. Let $q \le +\infty$. Suppose that for every connected component Y of the zero set $\{f = 0\}$ the germ f_Y is a sum of q squares of meromorphic function germs with controlled bad set. Then there exist an analytic function $g : \mathbb{R}^n \to \mathbb{R}$ with $\{g = 0\} \subset \{f = 0\}$, such that $g^2 f$ divides a sum $\sum_{i=1}^q a_i^2$ of q squares of analytic functions on $\mathbb{R}^n$ and the analytic function $\sum_{i=1}^q a_i^2/g^2 f$ is strictly positive in a neighborhood of $\{f = 0\}$.*

Proof. We will split the proof into several steps.

Step 1: Preparation. Let $\{Y_i\}$ denote the connected components of $\{f = 0\}$, and consider an open neighbourhood U of $\mathbb{R}^n$ in $\mathbb{C}^n$ on which f has an invariant holomorphic extension F. By hypothesis, for each i there are invariant holomorphic functions $G_i, B_{ik} : U_i \to \mathbb{C}$, defined on an open neighbourhood $U_i \subset U$ of Y_i in $\mathbb{C}^n$, such that $G_i^2 F|_{U_i} = \sum_k B_{ik}^2$, the series converges in the strong sense of 1.2*(ii)*, and $G_i^{-1}(0) \cap \mathbb{R}^n \subset Y_i$. Since the Y_i's form a locally finite family of disjoint closed subsets of $\mathbb{R}^n$, the U_i's may be chosen to form a locally finite family of disjoint open subsets of $\mathbb{C}^n$; next, take shrinkings $U_i' \subset \mathrm{Cl}_{\mathbb{C}^n}(U_i') \subset U_i$. Denote

$$D_i = U_i' \cap G_i^{-1}(0), \quad E_i = \mathrm{Cl}_{\mathbb{C}^n}(D_i) \setminus D_i.$$

Since $\mathrm{Cl}_{\mathbb{C}^n}(D_i) \subset G_i^{-1}(0)$:

$$E_i \cap \mathbb{R}^n \subset (G_i^{-1}(0) \setminus D_i) \cap \mathbb{R}^n = (G_i^{-1}(0) \setminus U_i') \cap \mathbb{R}^n \subset Y_i \setminus U_i' = \emptyset.$$

Each D_i is locally closed in $\mathbb{C}^n$, hence open in its closure $\mathrm{Cl}_{\mathbb{C}^n}(D_i) \subset \mathrm{Cl}_{\mathbb{C}^n}(U_i') \subset U_i$. We conclude that the boundaries E_i form a locally finite familly of closed subsets of $\mathbb{C}^n$, and their union E is a closed subset of $\mathbb{C}^n$. By the preceding remark, $E \cap \mathbb{R}^n = \emptyset$. Consequently, the open set

$$U' = \big(U \setminus (F^{-1}(0) \cup E)\big) \cup \bigcup_i U_i'$$

contains $\mathbb{R}^n$, and $D = \bigcup_i D_i$ is a closed analytic subset of U' (indeed, E is the boundary of D).

Next, take an invariant open Stein neighborhood $\mathcal{U} \subset U'$ of $\mathbb{R}^n$ in $\mathbb{C}^n$, such that $\mathbb{R}^n$ is a deformation retract of $\mathcal{U}$ ([Ca]). We denote

$$V_i = U'_i \cap \mathcal{U}, \quad T_i = D_i \cap \mathcal{U},$$

and keep F for the restriction of F to $\mathcal{U}$, and G_i, B_{ik} for those of G_i, B_{ik} to V_i. It holds:

- the connected component S_i of $F^{-1}(0)$ that contains Y_i is contained in V_i
- all T_i's are closed analytic subsets of $\mathcal{U}$, as well as their union $T = \bigcup_i T_i$.

Step 2: Glueing of denominators. After the preceding preparation, we glue the denominators G_i, each defined only in the corresponding V_i, to find a global common one G defined on the whole of $\mathcal{U}$.

Consider the coherent sheaf of ideals $\mathcal{J}$ defined on $\mathcal{U}$ by

$$\mathcal{J}_x = \begin{cases} G_i \, \mathcal{O}_{\mathbb{C}^n,x} & \text{if} \quad x \in T_i \\ \mathcal{O}_{\mathbb{C}^n,x} & \text{if} \quad x \notin T. \end{cases}$$

As was did before, the locally principal coherent sheaf $\mathcal{J}$ is globally principal, say generated by an holomorphic function $\Gamma : \mathcal{U} \to \mathbb{C}$, whose zero set is $\Gamma^{-1}(0) = T$. As the G_i's are invariant, $G = \Gamma \cdot \overline{\Gamma \circ \sigma}$ is an invariant holomorphic function on $\mathcal{U}$ that generates $\mathcal{J}^2$ at every real point $x \in \mathbb{R}^n$ of $\mathcal{U}$. Consequently, shrinking $\mathcal{U}$ we may assume G generates $\mathcal{J}^2$. Consider also the real analytic function $g = G|_{\mathbb{R}^n}$. The zero set of G is $T = \bigcup_i T_i$ and the zero set of g is $\bigcup_i T_i \cap \mathbb{R}^n \subset \bigcup_i Y_i \subset f^{-1}(0)$.

Next, $g^2 f$ is the restriction of $F' = G^2 F$ to $\mathbb{R}^n$. The zero set of $g^2 f$ is that of f, and its connected components are the Y_i's. By 2.2, after shrinking $\mathcal{U}$, the connected components $\{S_i\}_i$ of $(F')^{-1}(0)$ that intersect $\mathbb{R}^n$ can be numbered so that $S_i \cap \mathbb{R}^n = Y_i$; note also that $S_i \subset V_i$.

Now, since G generates $\mathcal{J}^2$, G_i^2 generates $\mathcal{J}^2|_{V_i}$, and these functions are invariant, there exist an invariant holomorphic function $Q_i : V_i \to C$ such that $G|_{V_i} = Q_i G_i^2$. We deduce:

$$F' = Q_i^2 G_i^2 (G_i^2 F) = Q_i^2 G_i^2 \sum_k B_{ik}^2 = \sum_k C_{ik}^2,$$

where $C_{ik} = Q_i G_i B_{ik}$, and the series $\sum_k C_{ik}^2$ verifies the convergence condition 1.2*(ii)*.

Step 3: Globalization of sums of squares. Here we find global sums of squares $\sum_k A_{ik}^2$ to replace the sums $\sum_k C_{ik}^2$, which are defined only on the V_i's.

First, up to shrinking V_i, we may assume that it is invariant and does not intersect any connected component of ${F'}^{-1}(0)$ other than S_i. By 2.4, applied to $\Phi = {F'}^2$, $V = V_i$ and $C_k = C_{ik}$, there exist invariant holomorphic functions $A_{ik} : \mathcal{U} \to \mathbb{C}$, such that $\sum_k \sup_K |A_{ik}|^2 < +\infty$ for all compact set $K \subset \mathcal{U}$ and ${F'}^2$ divides $A_{ik} - C_{ik}$ on V_i.

On V_i we have:

$$\sum_k A_{ik}^2 - F' = \sum_k A_{ik}^2 - \sum_k C_{ik}^2 = \sum_k \left(A_{ik}^2 - C_{ik}^2\right),$$

and this series is convergent on compact sets, as $\sum_k A_{ik}^2$ and $\sum_k C_{ik}^2$ are so. By construction, ${F'}^2$ divides on V_i each term $A_{ik}^2 - C_{ik}^2 = (A_{ik} + C_{ik})(A_{ik} - C_{ik})$, hence it divides their sum $\sum_k A_{ik}^2 - F'$. Thus there is an holomorphic function $\Psi_i : V_i \to \mathbb{C}$ such that on V_i we have:

$$\sum_k A_{ik}^2 = F' + \Psi_i {F'}^2 = u_i F', \quad \text{where } u_i = 1 + \Psi_i F'.$$

Clearly, u_i has no zeros in S_i, hence, u_i is a holomorphic unit in a perhaps smaller V_i.

Step 4: Auxiliary construction. We need a further refinement of the coverings involved so far.

Let $\{K_i\}_{i\geq 1}$ be a family of invariant compact subsets of $\mathcal{U}$ such that $K_i \cap \mathbb{R}^n \neq \emptyset$, $K_i \subset \mathrm{Int}_{\mathbb{C}^n}(K_{i+1})$ for all i, and $\bigcup_i K_i = \mathcal{U}$. Since the S_i's form a locally finite family in $\mathcal{U}$, we may extract a subfamily of these K_i's that in addition verifies $S_i \cap K_i = \emptyset$ for all i.

Next, by 2.2 applied to F', there exist holomorphic functions $\Lambda_i : \mathcal{U} \to \mathbb{C}$ such that

- $\Lambda_i^{-1}(0) = S_i$, and S_i is the connected component of $(F' + \Lambda_i^2)^{-1}(0)$ that contains Y_i,
- there is a holomorphic unit v_i defined on an open neighborhood of S, which we may suppose to be V_i, such that $F' + \Lambda_i^2 = v_i F'$.
- $F' + \Lambda_i^2$ has no zero in K_i, and
- $\lambda_i = \Lambda_i|_{\mathbb{R}^n}$ is a positive semidefinite real analytic function.

We deduce that the real zeros of $F' + \Lambda_i^2$ are contained in Y_i, hence the connected components of $\{F' + \Lambda_i^2 = 0\}$ other than S_i do not meet $\mathbb{R}^n$, and dropping them, we get an open neighborhood W_i of $K_i \cup \mathbb{R}^n$ on which

$$w_i = \frac{F'}{F' + \Lambda_i^2}$$

is holomorphic, and $(F' + \Lambda_i^2)^{-1}(0) \cap W_i = S_i$. As a matter of fact, *there is a common open neighborhood $W \subset \mathcal{U}$ of $\mathbb{R}^n$ on which all the above quotients w_i are holomorphic, and $(F' + \Lambda_i^2)^{-1}(0) \cap W \subset S_i$.*

Indeed, it is enough to find for each $x \in \mathbb{R}^n$, an open neighborhood W^x on which the required properties hold true, and the union of these W^x's will be the W we seek. But $x \in \mathrm{Int}_{\mathbb{C}^n}(K_{i_0})$ for some i_0, hence $x \in K_i$ for all $i \geq i_0$. Consequently all the quotients are holomorphic in $W^x = W_1 \cap \cdots \cap W_{i_0-1} \cap \mathrm{Int}_{\mathbb{C}^n}(K_{i_0})$, and if $z \in W^x$ is a zero of $F' + \Lambda_i^2$, then $i < i_0$, hence $z \in W_i$ and $z \in S_i$.

Step 5: Glueing of sums of squares. Here we paste all the sums of squares $\sum_k A_{ik}^2$ to get a single one.

For each i set

$$\mu_i = \sup_{K_i} \left| \frac{F'}{F' + \Lambda_i^2} \right|^2 \cdot \sum_k \sup_{K_i} |A_{ik}^2| \quad \text{and} \quad \gamma_i = \frac{1}{\sqrt{2^i \mu_i}}$$

We have

$$\sum_k \sup_{K_i} \left| \gamma_i \frac{F'}{F' + \Lambda_i^2} A_{ik} \right|^2 \leq \gamma_i^2 \sup_{K_i} \left| \frac{F'}{F' + \Lambda_i^2} \right|^2 \sum_k \sup_{K_i} |A_{ik}^2| \leq \frac{1}{2^i} \, .$$

Now, let K a compact subset of the open set W on which all the functions

$$\gamma_i^2 \left(\frac{F'}{F' + \Lambda_i^2} \right)^2 A_{ik}^2$$

are holomorphic. As $W \subset \bigcup_{i \geq 1} \mathrm{Int}_{\mathbb{C}^n}(K_i)$, K is contained in some K_{i_0}, hence in all K_i for $i \geq i_0$, and so:

$$\begin{aligned}
&\sum_{i,k} \sup_K \left| \gamma_i \frac{F'}{F' + \Lambda_i^2} A_{ik} \right|^2 \\
&\quad = \sum_{i=1}^{i_0 - 1} \sum_k \sup_K \left| \gamma_i \frac{F'}{F' + \Lambda_i^2} A_{ik} \right|^2 + \sum_{i \geq i_0} \sum_k \sup_K \left| \gamma_i \frac{F'}{F' + \Lambda_i^2} A_{ik} \right|^2 \\
&\quad \leq \sum_{i=1}^{i_0 - 1} \sup_K \left| \gamma_i \frac{F'}{F' + \Lambda_i^2} \right|^2 \sum_k \sup_K |A_{ik}^2| + \sum_{i \geq i_0} \sum_k \sup_{K_i} \left| \gamma_i \frac{F'}{F' + \Lambda_i^2} A_{ik} \right|^2 \\
&\quad \leq \sum_{i=1}^{i_0 - 1} \sup_K \left| \gamma_i \frac{F'}{F' + \Lambda_i^2} \right|^2 \sum_k \sup_K |A_{ik}^2| + \sum_{i \geq i_0} \frac{1}{2^i} \\
&\quad \leq \sum_{i=1}^{i_0 - 1} \sup_K \left| \gamma_i \frac{F'}{F' + \Lambda_i^2} \right|^2 \sum_k \sup_K |A_{ik}^2| + 1 < +\infty.
\end{aligned}$$

Consequently, the sum of squares $F'' = \sum_{i,k} \left(\gamma_i \frac{F'}{F' + \Lambda_i^2} A_{ik} \right)^2$ is convergent in the sense of 1.2*(ii)*.

Fix now i. On $W \cap V_i$ all $F' + \Lambda_j^2$, $j \neq i$, are units, and we can write

$$F'' = \left(\frac{\gamma_i F'}{F' + \Lambda_i^2} \right)^2 \sum_k A_{ik}^2 + \sum_{j \neq i} \left(\frac{\gamma_j F'}{F' + \Lambda_j^2} \right)^2 \sum_k A_{jk}^2 = \left(\frac{\gamma_i F'}{F' + \Lambda_i^2} \right)^2 \sum_k A_{ik}^2 + \Delta_i F'^2$$

where Δ_i is a holomorphic function. Here, we recall that for suitable holomorphic units u_i, w_i, on V_i we have:

$$\sum_k A_{ik}^2 = u_i F' \quad \text{and} \quad w_i = \frac{F'}{F' + \Lambda_i^2},$$

so that:

$$F'' = (\gamma_i^2 w_i^2 u_i + \Delta_i F')F'$$

But clearly the holomorphic function $\gamma_i^2 w_i^2 u_i + \Delta_i F'$ has no zeros in $W \cap S_i$, and consequently, is a holomorphic unit on $W \cap V_i$ (maybe after shrinking the neighborhood V_i of S_i).

This shows that the meromorphic function F''/F' is a holomorphic unit in a neighborhood of $\bigcup_i S_i$.

Step 6: Counting the number of squares. If we are dealing with infinite sums of squares, the proof ends here, but in case $q < +\infty$, we do not obtain a finite sum of squares. To amend this we must modify the argument a little.

Define, for each i:

$$M_i = \sup \left\{ \sup_{K_i} \left| \frac{F'}{F' + \Lambda_i^2} \right|^2 |A_{i1}|, \ldots, \sup_{K_i} \left| \frac{F'}{F' + \Lambda_i^2} \right|^2 |A_{iq}| \right\}.$$

and $\gamma_i = \frac{1}{2^i M_i}$. On K_i we have

$$\left| \gamma_i \left(\frac{F'}{F' + \Lambda_i^2} \right)^2 A_{ik} \right| \leq \frac{1}{2^i}\,.$$

Then, arguing as in the preceding step, one sees that each infinite sum

$$A_k = \sum_i \gamma_i \left(\frac{F'}{F' + \Lambda_i^2} \right)^2 A_{ik}, \qquad k = 1, \ldots, q,$$

is a well defined holomorphic function on a neighborhood $W \subset \mathcal{U}$ of $\mathbb{R}^n$, and then one proceeds as follows. On $W \cap V_i$ one writes

$$A_k = \gamma_i \left(\frac{F'}{F' + \Lambda_i^2} \right)^2 A_{ik} + \sum_{j \neq i} \gamma_j \left(\frac{F'}{F' + \Lambda_j^2} \right)^2 A_{jk} = \gamma_i w_i^2 A_{ik} + \Delta_{ik} F'^2$$

where Δ_{ik} is a holomorphic function. Hence

$$F'' = \sum_{k=1}^{q} A_k^2 = \gamma_i^2 w_i^4 \sum_{k=1}^{q} A_{ik}^2 + \Delta_i F'^2 = (\gamma_i^2 w_i^4 u_i + \Delta_i F')F',$$

where $\Delta_i = \sum_k 2\gamma_i w_i^2 A_{ik} \Delta_{ik} + \sum_k \Delta_{ik}^2 F'^2$. Thus

$$F'' = (\gamma_i^2 w_i^4 u_i + \Delta_i F')F'$$

and the proof ends as in the preceding step. $\square$

Remark 3.2 The preceding proof consists esentially in modifying several local representations as sums of squares to get a global one. Thus, if a part of those local representations is already global, one can expect to keep it stable under the construction. This is indeed the case, and we can add to the statement 3.1 the following paragraph:

Suppose that there is a global analytic function $h : \mathbb{R}^n \to \mathbb{R}$ such that every germ f_Y is a sum of q squares of meromorphic function germs, the first one divisible by h_Z. Then a_1 is divisible by h.

To prove this, one reads the proof from the very beginning, tracking the construction of the A_{ik}'s. By hypothesis, we now have in *Step 1* holomorphic functions $B^*_{i1} : V_i \to \mathbb{C}$ such that $B_{i1} = B^*_{i1} H$, where H is an holomorphic extension of h. Thus, in *Step 2* we get $C_{i1} = Q_i G_i B^*_{i1} H$. Next, in *Step 3,* we use 2.5 to guarantee that $A_{i1} = A^*_{i1} H$, with A^*_{i1} holomorphic on $\mathcal{U}$. Thus in *Step 5* all terms A_{i1} are divisible by H, and any one can be chosen as the first term. Finally, in *Step 6* the first term is:

$$A_1 = \sum_i \gamma_i \left(\frac{F'}{F' + \Lambda_i^2} \right)^2 A_{i1} = \left(\sum_i \gamma_i \left(\frac{F'}{F' + \Lambda_i^2} \right)^2 A^*_{i1} \right) H.$$

□

4 Proof of the main theorem

We are ready for the:

Proof of Theorem 1.4. We choose an analytic equation $h : \mathbb{R}^n \to \mathbb{R}$, that is: $M = \{h = 0\}$. Consider a tubular neighborhood Ω of M in $\mathbb{R}^n$, endowed with the corresponding analytic retraction $\pi : \Omega \to M$. As usual, by composition with π, all functions extend from M to Ω; we will denote the extensions with bars. In particular $f : M \to \mathbb{R}$ extends to a positive semidefinite analytic function $\bar{f} : \Omega \to \mathbb{R}$. Surely, the zero set of $\bar{f}$ extends off M, but we take $f' = h^2 + \bar{f}$, so that $X = \{f' = 0\} = \{f = 0\} \subset M$. Let Y be a connected component of X. On a neighborhood of Y in M we have a representation as a sum of q squares:

$$g_i^2 f = \sum_k b_{ik}^2,$$

with $\{g_i = 0\} \subset \{f = 0\}$. After composition with π, this representation extends to a neighborhood of Y in Ω, hence in $\mathbb{R}^n$, and we can write there:

$$\bar{g}_i^2 f' = \bar{g}_i^2 h^2 + \bar{g}_i^2 \bar{f} = \bar{g}_i^2 h^2 + \sum_k \bar{b}_{ik}^2.$$

Thus we have a representation of f'_Y as a sum of $q + 1$ squares, the first one divisible by h. Finally, it remains to extend f' to $\mathbb{R}^n$. Once again, we consider a locally principal sheaf, namely:

$$\mathcal{J}_x = \begin{cases} f'_x\,\mathcal{O}_{\mathbb{R}^n,x} & \text{if} \quad x \in U \\ \mathcal{O}_{\mathbb{R}^n,x} & \text{if} \quad x \notin X. \end{cases}$$

As $H^1(\mathbb{R}^n, \mathcal{O}^*_{\mathbb{R}^n}) = H^1(\mathbb{R}^n, \mathbb{Z}_2) = 0$, this sheaf is globally principal, say generated by f''. The zero set of f'' is X, and $f' = vf''$, where v is an analytic unit on Ω. Clearly. the sign of f'' is locally constant, hence constant; we deduce that the sign of v is constant too, and maybe replacing f'' by $-f''$, f'' is positive semidefinite on $\mathbb{R}^n$ and $v > 0$ on Ω; set $w = +\sqrt{v}$, which is an analytic unit on Ω.

After this preparation, Theorem 3.1 applies to f'', and we find analytic functions $g, u : \mathbb{R}^n \to \mathbb{R}$, $\{g = 0\} \subset \{f'' = 0\}$ and u strictly positive in a neighborhood of $\{f'' = 0\}$, such that $ug^2 f'' = \sum_{i=1}^{q+1} a_i^2$ for some analytic functions a_i. Furthermore, by Remark 3.2, the first square a_1 is divisible by h. The only trouble is whether or not the sum of squares $\sum_{i=1}^{q+1} a_i^2$ has zeros off $\{f'' = 0\}$, but we can fix it with an additional square.

Indeed,

$$\alpha = \frac{g^4 f''^2 + \sum_{i=1}^{q+1} a_i^2}{g^2 f''} = g^2 f'' + u$$

is a well defined strictly positive analytic function: both addends in the right hand side are ≥ 0, the first one does not vanish off $\{f'' = 0\}$, and the second one does not vanish on $\{f'' = 0\}$. Now, let β stand for the positive square root of α, and we have:

$$\beta^2 g^2 f'' = g^4 f''^2 + \sum_{i=1}^{q+1} a_i^2,$$

where h divides a_1. Finally, when we restrict everything to M, the square a_1^2 disappears, and we find that $f''|_M$ is a sum of $q+1$ squares of meromorphic functions with controlled bad set. But $f = w^2|_M f''|_M$, where w has no zero in M, and we are done. $\square$

The preceding theorem can be sometimes combined with the standard trick that factors out the codim 1 components of the zero set of a positive semidefinite analytic function. We record this trick here for later use. As mentioned in the introduction, we denote by r the smallest integer such that all locally principal analytic sheaves are generated by r global analytic functions. It is known that $r \leq m+1$ (see [Co]); the fact that $r = 1$ for $M = \mathbb{R}^m$ has already been used and will again be essential later.

Lemma 4.1 Let M be a real analytic manifold and r as above. Let $f : M \to \mathbb{R}$ be a positive semidefinite analytic function. Then we can factorize $f = \big(h_1^2 + \cdots + h_r^2\big) f'$, where $h_1, \ldots, h_r, f'$ are analytic functions on M such that the zero set $\{f' = 0\}$ of the latter has codimension ≥ 2.

Proof. Firstly, at each zero x of f, we write $f_x = \zeta_x^2 \eta_x \in \mathcal{O}_{M,x}$, η_x without multiple factors; this factorization is unique up to units. The germ $\{\eta_x = 0\}$ has codimension ≥ 2, since otherwise some irreducible factor ξ_x of η_x would be real, and f_x would change sign at x.

Now, the ζ_x's generate a locally principal coherent sheaf $\mathcal{J} \subset \mathcal{O}_M$. Thus, $\mathcal{J}$ is globally generated by r analytic functions $h_1, \ldots, h_r : M \to \mathbb{R}$. An easy computation shows that $h_1^2 + \cdots + h_r^2$ generates the sheaf $\mathcal{J}^2$, and we have $f = \big(h_1^2 + \cdots + h_r^2\big) f'$. Each germ f'_x coincides with η_x up to a unit, hence its zero set has codimension ≥ 2 and f'_x does not change sign. We are done. □

It is clear that to represent f in the statement above as a sum of squares it is enough to represent f'. Hence, we can apply Theorem 1.4 to f', whose zero set has codim ≥ 2. Note that in the end, f will be represented by r times as many squares as f'. For instance, jointly with Proposition 1.5, this gives the following modifications of Corollary 1.9 and Corollary 1.12:

Corollary 4.2 *If* h17 *holds for all global analytic sets $Y \subset M$ of codimension ≥ 2, then it holds for M.*

Corollary 4.3 *If there is an integer p such that $p_Y \leq p$ for all global analytic sets $Y \subset M$ of codimension ≥ 2, then $p(M) = p_M \leq r(2^{n-1}p + 1)$.*

5 Bad sets

The purpose of this section is to show how to control the bad set of a sum of squares of meromorphic functions, which is the content of 1.5. This control is essential to apply Theorem 1.4. First of all, we can always reduce to the case when $M \subset \mathbb{R}^n$ is an open set $\Omega \subset \mathbb{R}^n$. For, given a tubular neighborhood Ω of M, there is an analytic retraction $\pi : \Omega \to M$, and via π we can pull back all data on M to data on Ω. This respects convergence conditions, as one easily checks by complexifying the retraction π.

After this remark, it is clear that the following statement implies Proposition 1.5:

Lemma 5.1 *Let $f : \Omega \to \mathbb{R}$ be an analytic function defined on an open set $\Omega \subset \mathbb{R}^n$. Let $h : \Omega \to \mathbb{R}$ be an analytic function such that $h^2 f$ is a sum of $p \leq +\infty$ squares of analytic functions. Set* $\dim\{h = 0, f \neq 0\} = d$. *Then, there exist an analytic function $g : \Omega \to \mathbb{R}$ such that $g^2 f$ is a sum of $q \leq 2^{d+1}p$ squares, and $\{g = 0\} \subset \{f = 0\}$. Moreover, on a smaller neighborhood of $\{f = 0\}$ we can assume $r \leq 2^d p$.*

Proof. Consider the global analytic set $Y = \{h = 0\}$. We pick a point y_i in each irreducible component Y_i of Y that is not contained in $\{f = 0\}$. Clearly, we can suppose $f(y_i) \neq 0$ and that the y_i's form a discrete set. By a small diffeotopy around each y_i we can move y_i off Y, to obtain a smooth diffeomorphism $\psi : \Omega \to \Omega$ which maps each y_i to $y'_i \notin Y$ and is the identity on a neighborhood of $\{f = 0\}$. By the latter condition, f^2 divides the map $\psi - \mathrm{Id}$, hence $\psi = \mathrm{Id} + f^2\mu$ for a smooth map $\mu : \Omega \to \mathbb{R}^n$. Now, let $\eta : \Omega \to \mathbb{R}^n$ be an analytic mapping close to μ. Then $\varphi = \mathrm{Id} + f^2\eta$ is close to ψ, and consequently φ is an analytic diffeomorphism of Ω. Note that φ is the identity on $\{f = 0\}$, and so f and $f \circ \varphi$ have the same zeros. Also, by looking at Taylor expansions, one sees that $f \circ \varphi = f + f^2 h$ for some analytic map $h : \Omega \to \mathbb{R}^n$. Thus we can write $f \circ \varphi = vf$, where $v = 1 + fh$ does not have zeros: a zero x of v would be a zero of $f \circ \varphi$, hence one of f, and $v(x) = 1 + f(x)h(x) = 1$! Moreover, as f is positive semidefinite, so is v, and $u = \sqrt{v}$ is a well defined strictly positive analytic function such that $f \circ \varphi = u^2 f$. By hypothesis $h^2 f = \sum_j h_j^2$, which gives:

$$u^2(h \circ \varphi)^2 f = (h \circ \varphi)^2 (f \circ \varphi) = \sum_j (h_j \circ \varphi)^2$$

(note that if the sum is infinite, it is well defined in the sense of 1.2). Hence,

$$(h^2 + u^2(h \circ \varphi)^2) f = \sum_j (h_j^2 + (h_j \circ \varphi)^2).$$

Now, we multiply both sides times $h^2 + u^2(h \circ \varphi)^2$ to get

$$\delta^2 f = \sum_j (h_j^2 + (h_j \circ \varphi)^2)(h^2 + u^2(h \circ \varphi)^2),$$

with $\delta = h^2 + u^2(h \circ \varphi)^2$. If the sum is infinite, we have another infinite sum. In case the sum is finite, then we recall that the product of two sums of two squares is again a sum of two squares, and we get twice the number of squares. Finally, the bad set now is:

$$\{\delta = 0\} = \{h = 0\} \cap \{h \circ \varphi = 0\} = \{h = 0\} \cap \varphi^{-1}(Y),$$

so that,

$$\{\delta = 0\} \setminus \{f = 0\} \subset \bigcup_i Y_i \cap \varphi^{-1}(Y).$$

But no irreducible component Y_i is contained in $\varphi^{-1}(Y)$, because $\varphi(y_i) \notin Y$, hence $\dim(Y_i \cap \varphi^{-1}(Y)) < \dim Y_i \leq d$.

Thus we drop the dimension of the bad set off $\{f = 0\}$, and after $d+1$ repetitions we get the first assertion of the statement. Instead, we can stop after d times, and then

$$\dim\{g = 0, f \neq 0\} \leq 0.$$

This means that $D = \{g = 0, f \neq 0\}$ is a discrete closed subset of Ω, and this latter can be replaced by $\Omega \setminus D$ to get the second assertion. □

The control of bad sets is important when dealing with restrictions of sums of squares of meromorphic functions. The problem is that in such a restriction the denominator can vanish identically, and then we are left with nothing. An interesting example of this control is the following:

Lemma 5.2 *Let $M_2 \subset M_1 \subset \mathbb{R}^l$ be closed manifolds. If* h17 *holds for M_1, then it holds for M_2.*

Proof. We must show that every positive semidefinite analytic function $f : M_2 \to \mathbb{R}$ is a sum of squares.. We pick a global equation $h : \mathbb{R}^l \to \mathbb{R}$ of M_2, that is, an analytic function on $\mathbb{R}^l$ such that $M_2 = \{h = 0\}$, and a tubular neighborhood U of M_2 in $\mathbb{R}^l$, equiped with the corresponding retraction $\pi : U \to M_2$. As we did in the proof of Theorem 1.4, the analytic function $g = f \circ \pi + h^2 : U \to \mathbb{R}$ is positive semidefinite, and extends f. Furthermore, $\{g = 0\} = \{f = 0\}$, which is closed in M_2, hence in $\mathbb{R}^l$. Again as in the proof of Theorem 1.4, this enables us to extend g to a positive semidefinite analytic function $\bar{g}$ on $\mathbb{R}^l$: there is an analytic unit $u : U \to \mathbb{R}$ such that $\bar{g} = u^2 g$ on U. Then we consider $\widetilde{g} = \bar{g}|_{M_1}$, and since h17 holds for M_1, $\widetilde{g}$ is a sum of squares of meromorphic functions:

$$h^2 \widetilde{g} = \sum_k a_k^2 \qquad (*)$$

In addition, we can control the bad set, and, by Lemma 1.5, suppose $\{h = 0\} \subset \{\widetilde{g} = 0\}$. By this control, the restriction of $(*)$ to M_2 is well defined:

$$\{h = 0\} \cap M_2 \subset \{\widetilde{g} = 0\} \cap M_2 = \{f = 0\}$$

is a proper subset of M_2, and so the denominator h does not vanish identically on M_2. Finally, $\widetilde{g}|M_2 = u^2|_{M_2} f$, where u has no zero in M_2, and we are done. □

Using this lemma we get:

Proof of Proposition 1.10(a). Suppose that h17 holds for $M \times \mathbb{R}$, and let Z be a closed subset of M. We must show that every positive semidefinite analytic function $f : W \to \mathbb{R}$ defined on a neighborhood of Z is a sum of squares of meromorphic functions, perhaps after shrinking W. We can assume that $W = \{\theta > 0\}$ for some analytic function $\theta : M \to \mathbb{R}$ (pick a close analytic approximation of any smooth function $\geq \frac{1}{2}$ on Z and $\leq -\frac{1}{2}$ on $M \setminus W$, see [Na]). In this situation, we consider the analytic embedding

$$\varphi : W \to M \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R} : x \mapsto \Big(x, \frac{1}{\theta(x)}\Big).$$

Clearly, $N = \varphi(W) \subset M \times \mathbb{R} \subset \mathbb{R}^{n+1}$ are closed analytic manifolds, and h17 holds for $M \times \mathbb{R}$, hence, by the lemma, it holds for N. Consequently, $f' = f \circ \varphi^{-1} : N \to \mathbb{R}$ is a sum of squares of meromorphic functions, and so is f as wanted. □

Proof of Proposition 1.10(b). Suppose that h17 holds for $M \times \mathbb{R}$. We must show that every positive semidefinite analytic function $f : W \to \mathbb{R}$ defined on a neighborhood $W \subset \mathbb{R}^m$ of a closed set $Z \subset \mathbb{R}^m$ is a sum of squares of meromorphic functions, perhaps after shrinking W.

We can assume that $W = \{\theta > 0\}$ for some analytic function $\theta : \mathbb{R}^m \to \mathbb{R}$ which vanishes at infinity:

$$\lim_{\|x\| \to \infty} \theta(x) = 0$$

(replace θ by $\frac{\theta}{\eta(1+\theta^2)}$, where $\eta(x) = e^{\|x\|^2}$). Now, pick any open set $U \subset M$ analytically diffeomorphic to $\mathbb{R}^m$, so that we can identify $U \equiv \mathbb{R}^m \supset W \supset Z$. In this situation, we consider the analytic embedding

$$\varphi : W \to M \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R} : x \mapsto \Big(x, \frac{1}{\theta(x)}\Big).$$

By the condition of θ at infinity, the image $N = \varphi(W)$ is closed in $\mathbb{R}^{n+1}$.

Indeed, let $(x, t) \to (x_0, t_0)$, with $(x, t) \in N$. Then, $\frac{1}{\theta(x)} \to t_0 \in \mathbb{R}$, so that $\theta(x) \to \theta_0 \neq 0$, which implies that $x \to x_0 \in U$. Hence

$$\theta(x_0) = \lim \theta(x) \geq 0,$$

because $x \in W = \{\theta > 0\}$. Consequently, $\theta_0 = \theta(x_0) > 0$, and $x_0 \in W$. We conclude:

$$t_0 = \lim \frac{1}{\theta(x)} = \frac{1}{\theta(x_0)},$$

so that $(x_0, t_0) \in N$.

Thus, $N = \varphi(W) \subset M \times \mathbb{R} \subset \mathbb{R}^{n+1}$ are closed analytic manifolds, and h17 holds for $M \times \mathbb{R}$, hence, by the lemma, it holds for N. Consequently, $f' = f \circ \varphi^{-1} : N \to \mathbb{R}$ is a sum of squares of meromorphic functions, and so is f. □

6 The finiteness implications of Hilbert's 17th Problem

In this section we discuss the finiteness nature of Hilbert's 17 Problem. We will need the following lemma:

Lemma 6.1 *Let $M_2 \subset M_1 \subset \mathbb{R}^l$ be closed manifolds.*

(a) Let $\sum_k f_k^2$ be a sum of squares of analytic functions $f_k : M_2 \to \mathbb{R}$. Then there is a sum of squares $\sum_k {h_k}^2$ of analytic functions $h_k : M_1 \to \mathbb{R}$ such that $h_k|_{M_2} = f_k$.

(b) Finiteness for M_1 implies finiteness for M_2.

(c) $p(M_2) \leq 2^l p(M_1) + 1$.

Proof. *(a)* Since every analytic function f on M_2 has some extension h to M_1 (Cartan's Theorem B), the problem is about convergence when the sum is infinite. This can be fixed as follows. Pick a tubular neighborhood U of M_2 in $\mathbb{R}^l$, with the corresponding analytic retraction $\pi : U \to M_2$, and an analytic equation $\theta : \mathbb{R}^l \to \mathbb{R}$ of M_2, that is, $M_2 = \theta^{-1}(0)$. Now we consider: *(i)* a complexification $\Theta : \mathcal{U} \to \mathbb{C}$ of θ, defined on an invariant open Stein neighborhood $\mathcal{U}$ of $\mathbb{R}^l$ in $\mathbb{C}^l$, *(ii)* a complexification $\Pi : V \to \mathbb{C}^l$ of π, defined on an invariant open neighborhood V of U in $\mathbb{C}^l$, and *(iii)* complexifications $F_k : V \to \mathbb{C}$ of the f_k's, with the convergence conditions of Definition 1.2*(ii)*. All these complexifications are invariant, and after some topological shrinkings, we can suppose that V contains the connected components of $\{\Theta = 0\}$ that intersect $\mathbb{R}^n$ and V does not meet the others. Then, denote $C_k = F_k \circ \Pi$. Clearly, the sum $\sum_k C_k^2$ is convergent as $\sum_k F_k^2$ is, and we can apply Lemma 2.4 with $\Phi = \Theta$, to get invariant holomorphic functions $A_k : \mathcal{U} \to \mathbb{C}$, such that the sum of squares $\sum_k A_k^2$ converges too, and $\Theta|_V$ divides all the differences $A_k|_V - C_k$. By this latter condition, A_k and C_k coincide on $\Theta^{-1}(0)$. Consider the real analytic functions $a_k = A_k|_{\mathbb{R}^l} : \mathbb{R}^l \to \mathbb{R}$, whose sum $\sum_k a_k^2$ is well defined. Furthermore, since $\Pi|_U = \pi$ is the identity on M_2 and $\Theta|_{\mathbb{R}^l} = \theta$ vanishes on M_2, we have $a_k|_{M_2} = f_k$. We conclude by taking $h_k = a_k|_{M_1}$.

(b) Suppose finiteness for M_1 and let $f = \sum_k f_k^2$ be a sum of squares of analytic functions on M_2. By *(a)* there is a sum of squares $h = \sum_k {h_k}^2$ of analytic functions h_k on M_1 such that $h_k|_{M_2} = f_k$. By assumption, h is a finite sum of squares of meromorphic functions on M_1, and by Proposition 1.5, with controlled bad set. This implies that the denominator does not vanish on M_2, hence by restriction we get that $f = h|_{M_2}$ is a finite sum of squares of meromorphic functions on M_2.

(c) The same argument of *(b)*, applied to finite sums, gives the bound in *(c)*. □

Now, we prove Proposition 1.13, which can be rephrased as follows:

Proposition 6.2 *Finiteness for $M \times \mathbb{R}$ implies $p_Z < +\infty$ for all closed subsets $Z \subset M$.*

Proof. Let $Z \subset M$ be closed, and suppose that $p_Z = +\infty$. Then for each $p \geq 1$ there is a sum of squares $f_p = \sum_k h_k^2$ defined on a neighborhood W_p of Z, which is not a sum of p squares of meromorphic functions on W_p. We can suppose $W_p = \{\theta_p > 0\}$ for suitable analytic functions $\theta_p : M \to \mathbb{R}$, and also $W_{p+1} \subset W_p$ for all p, and $\theta_{p+1} < \theta_p$ on W_{p+1}. Now, we consider the closed embeddings

$$\varphi_p : W_p \to M \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R} : x \mapsto \Big(x, \frac{1}{\theta_p(x)} + p\Big).$$

The images $N_p = \varphi_p(W_p) \subset M \times [p, \to)$ are manifolds closed in $M \times \mathbb{R}$, which obviously form a locally finite family. Furthermore, the N_p's are disjoint.

Indeed, suppose there is $x \in W_p \subset W_q$, $p > q$, such that $\frac{1}{\theta_p(x)} + p = \frac{1}{\theta_q(x)} + q$. As $\theta_p(x) < \theta_q(x)$, that equality would imply $q > p$, a contradiction.

Thus the union N of the N_p's is a closed manifold, whose connected components are those of the N_p's. Hence we can define an analytic function $f : N \to \mathbb{R}^n$ whose restriction to each N_p is f_p. By Theorem 1.4, f is a sum of squares in N, and that sum cannot be finite because the number of squares we would need in N_p is at least p, for all p. But by hypothesis, finiteness holds for $M \times \mathbb{R}$, hence it should hold for N too. Contradiction. $\square$

We turn to the:

Proof of Proposition 1.14. Suppose $p = p(M \times \mathbb{R}) < +\infty$. Consider any finite sum of squares $f = \sum_k f_k^2$ defined on an open neighborhood W of a closed subset $Z \subset M$. As usual, we can embed W in $M \times \mathbb{R}$ as a closed manifold N, and $p(N) \leq 2^{n+1}p+1$ by Lemma 6.1. We deduce that f is a sum of $q \leq 2^{n+1}p+1$ squares of meromorphic functions. The same is true then for the germ f_Z, and we conclude that $p(M_Z) \leq 2^{n+1}p+1$, a bound independent from the set Z.$\square$

Next, we prove Proposition 1.15 in the following form:

Proposition 6.3 *Finiteness for $M \times \mathbb{R}$ implies $p_Z < +\infty$ with a common bound for all closed subsets $Z \subset \mathbb{R}^m$.*

Proof. Suppose by way of contradiction that for each p there is a sum of squares $f_p : W_p \to \mathbb{R}$ defined on an open neighborhood W_p of a closed set $Z_p \subset \mathbb{R}^m$ that is not a sum of p squares of meromorphic functions. As in the proof of Proposition 1.10*(b),* we have $W_p = \{\theta_p > 0\}$ for some analytic function $\theta_p : \mathbb{R}^m \to \mathbb{R}$ which vanishes at infinity. Now, we pick disjoint open sets $U_p \subset M$ analytically diffeomorphic to $\mathbb{R}^m$, so that we can identify $U_p \equiv \mathbb{R}^m \supset W_p \supset Z_p$. Then, the analytic embedding

$$\varphi_p : W_p \to M \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R} : x \mapsto \Big(x, \frac{1}{\theta_p(x)} + p\Big).$$

is closed. Now, as in the proof of Proposition 6.2, $N = \bigcup_p N_p \subset M \times \mathbb{R}$ is a closed manifold, on which the f_p's define well a sum of squares which is not finite. Thus finiteness fails for N, hence for $M \times \mathbb{R}$, by Lemma 6.1*(b).* $\square$

Next we will discuss the improvements of these results for $M = \mathbb{R}^m$. To start with, we prove the following equivalent version of the first half of Proposition 1.16.

Proposition 6.4 *Suppose that $p(\mathbb{R}^m) = +\infty$. Then, there is a positive semidefinite analytic function $f : \mathbb{R}^m \to \mathbb{R}$ that is a sum of squares of meromorphic functions, but not a finite sum of squares.*

Proof. If $p(\mathbb{R}^m) = +\infty$, for each $p \geq 1$ there is an analytic function $f_p : \mathbb{R}^m \to \mathbb{R}$ which is a sum of squares of meromorphic functions, but not of p squares. By Lemma 4.1, we may suppose that the zero set X_p of f_p has codimension 2. Assume for a moment that X_p can be moved into the open cylinder

$$V_p = \{x = (x', x_m) \in \mathbb{R}^m : \|x' - a'_p\| < \tfrac{1}{4}\},$$

where $a'_p = (p, 0, \dots, 0) \in \mathbb{R}^{m-1}$. Then the X_p's form a locally finite family, and $X = \bigcup_p X_p$ is a closed analytic subset of $\mathbb{R}^m$. Consequently, we can define the following locally principal sheaf :

$$\mathcal{J}_x = \begin{cases} f_p \mathcal{O}_{\mathbb{R}^m,x} & \text{if} \quad x \in X_p \\ \mathcal{O}_{\mathbb{R}^m,x} & \text{if} \quad x \notin X. \end{cases}$$

Since on $\mathbb{R}^m$ locally principal sheaves are globally principal, $\mathcal{J}$ has a global generator f. Thus, on each V_p there is an analytic unit v_p such that $f = v_p f_p$. Note also that the zero set of f is X, which does not disconnect $\mathbb{R}^m$, because its codimension is ≥ 2. Hence, f has constant sign on $\mathbb{R}^m$ and we may assume $f \geq 0$, and $v_p > 0$. In particular, $+\sqrt{v_p}$ is a well defined analytic function on V_p, so that f and f_p behave the same concerning sums of squares. Since, by construction, the connected components Y of X are the ones of the X_p's, we deduce that each germ f_Y is a sum of squares. Thus, by Theorem 1.4, f is a sum of squares of meromorphic functions. However, this sum cannot be finite, say of p squares, because f_p is not a sum of p squares.

To complete the proof it only remains to move each X_p by a suitable analytic diffeomorphism of $\mathbb{R}^m$. This we do now.

Since X_p has codimension ≥ 2, many lines do not meet X_p, and after a linear change of coordinates, we may assume this is the case for the x_m-axis. Then, we pick an analytic function $\delta(x_m)$ such that $0 < \delta(x_m) < \operatorname{dist}\big(X_p, (0, \dots, 0, x_m)\big)$, and the analytic diffeomorphism

$$(x', x_m) \mapsto \left(\frac{\sqrt{1 + x_m^2}}{\delta(x_m)}\, x', x_m \right)$$

moves X_p off $\{\|x'\|^2 < 1 + x_m^2\}$. Thus, we henceforth assume $X_p \subset \{\|x'\|^2 \geq 1 + x_m^2\}$. Then, we consider the analytic diffeomorphism: $\varphi(x', x_m) = (y', y_m)$ defined by the equations:

$$y' - a'_p = \frac{x'}{4(1 + y_m^2)}, \qquad y_m = 2\|x'\|^2 - x_m.$$

The conclusion is that $\varphi(X_p)$ is contained in $\|y' - a'_p\| < \frac{1}{4}$, and we are done.

Indeed, if $(x', x_m) \in X_p$, then $\|x'\|^2 \geq 1 + x_m^2$, so that:

$$y_m = 2\|x'\|^2 - x_m \geq \|x'\|^2 + 1 + x_m^2 - x_m > \|x'\|.$$

Consequently:

$$\|y' - a'_p\| = \frac{\|x'\|}{4(1 + y_m^2)} < \frac{\|x'\|}{4(1 + \|x'\|^2)} < \frac{1}{4}.$$

□

Next, we look at the Pythagoras number $p(\mathcal{M}_m)$ of the field $\mathcal{M}_m = \mathbb{R}(\{x_1, \dots, x_m\})$ of meromorphic power series. We reformulate the second assertion of Proposition 1.16 as follows:

Proposition 6.5 *Suppose that $p(\mathcal{M}_m) = +\infty$. Then, there is a positive semidefinite analytic function $f : \mathbb{R}^m \to \mathbb{R}$ that is a sum of squares of meromorphic functions, but not a finite sum of squares.*

Proof. Fix for each integer $p \geq 1$ a germ g_p which is a sum of squares of meromorphic function germs, but not of p squares. After a change of coordinates, in a suitable neighborhood $W \times I \subset \mathbb{R}^{m-1} \times \mathbb{R} = \mathbb{R}^m$ of the origin we have analytic functions $\delta, h_k : W \times I \to \mathbb{R}$ and $a_i : W \to \mathbb{R}$ with $a_i(0) = 0$, such that

$$g_p = x_m^d + a_1 x_m^{d-1} + \cdots + a_d = \sum_k h_k^2/\delta^2.$$

We can suppose $I = (-2\rho, 2\rho) \subset \mathbb{R}$, with $\rho > 0$, and we choose $\varepsilon > 0$ small enough so that

$$|a_1(x')\rho^{d-1} + \cdots + a_{d-1}(x')\rho + a_d(x')| < \rho^d \quad \text{for} \quad \|x'\| < \varepsilon.$$

We shrink W to $W = \{\|x'\| < \varepsilon\}$, for g_p to have not zeros in $W \times \{|x_m| = \rho\}$. Next, pick an analytic diffeomorphism

$$\varphi_p : U_p = \mathbb{R}^{m-1} \times (p - \tfrac{1}{2}, p + \tfrac{1}{2}) \to W \times I$$

which maps each level $x_m = p + t$ to the level $x_m = 4\rho t$. Set $f_p = g_p \circ \varphi_p$. The construction guarantees that alike g_p, the analytic function f_p is a sum of squares of meromorphic functions on U_p, but not of p squares.

Next we consider the open set $V_p = U_p \cap \{|x_m - p| < \frac{1}{4}\}$, and claim that $X_p = \{f_p = 0\} \cap V_p$ is closed in $\mathbb{R}^m$. For, suppose there is $x \notin X_p$ adherent to X_p, then $x \in U_p \cap \{|x_m - p| = \frac{1}{4}\}$, and $\varphi_p(x) \in W \times \{|x_m| = \rho\}$ is a zero of g_p, which is impossible.

By the claim, the union $X = \bigcup_p X_p$ is a closed analytic subset of $\mathbb{R}^m$, and we can define a coherent locally principal sheaf on $\mathbb{R}^m$ by

$$\mathcal{J}_x = \begin{cases} f_p \mathcal{O}_{\mathbb{R}^m,x} & \text{if} \quad x \in X_p \\ \mathcal{O}_{\mathbb{R}^m,x} & \text{if} \quad x \notin X \end{cases}$$

Once again, we know that $\mathcal{J}$ is globally principal, say generated by f. This is the function we sought.

Indeed, on each V_p there is an analytic unit v_p such that $f = v_p f_p$. Thus, the sign of f is locally constant, hence constant, and we can suppose $f \geq 0$, so that $v_p > 0$. Recall here that f_p is a sum of squares of meromorphic functions, hence f is also a sum of squares on V_p. On the other hand, the zero set of f is X, and its connected components Y are the connected components of the X_p's. Summing up, f verifies all conditions to apply once again Theorem 1.4, and we conclude that f is a sum of squares of meromorphic functions on $\mathbb{R}^m$. Finally, this sum cannot be finite, say of p squares, because then f_p would be a sum of p squares of meromorphic function germs, which we know is not the case. $\square$

7 Special results for functions with small zero sets

Here we will prove some better statements for positive semidefinite germs whose zero set has dimension ≤ 1. In this section, $M = \mathbb{R}^m$ and $m \geq 3$.

Proposition 7.1 *Let f_Z be an analytic germ at a closed set $Z \subset \mathbb{R}^m$, and suppose that its zero set germ $\{f_Z = 0\}$ has dimension ≤ 1. Then:*

(1) If h17 *holds for $\mathbb{R}^m$ and f_Z is positive semidefinite, then f_Z is a sum of $q \leq +\infty$ squares of meromorphic function germs.*

(2) If finiteness holds for $\mathbb{R}^m$ and f_Z is a sum of squares, then f_Z is a sum of $q \leq 2^{m-1}p(\mathbb{R}^m) + 1 < +\infty$ squares of meromorphic function germs with controlled bad set.

Combining these two facts, we see that *if* H17 *holds for $\mathbb{R}^m$ and f_Z is positive semidefinite with small zero set, then f_Z is a sum of $q < +\infty$ squares of meromorphic function germs, where q only depends on m.*

The proof of Proposition 7.1 relies on the following lemma:

Lemma 7.2 *Let $W \subset \mathbb{R}^m$ be an open set, $f : W \to \mathbb{R}$ a positive semidefinite analytic function such that* $\dim\{f = 0\} \leq 1$*, and let $Z \subset \mathbb{R}^m$ a closed set contained in W. Then, there are:*

(i) an open neighborhood $\Omega \subset W$ of $\{f = 0\} \cap Z$,

(ii) a positive semidefinite analytic function $g : \mathbb{R}^m \to \mathbb{R}$ with $\dim\{g = 0\} \leq 1$*,*

(iii) an analytic open embedding $\varphi : \Omega \to \mathbb{R}^m$, and

(iv) a strictly positive analytic function $u : \Omega \to \mathbb{R}$,

such that $g \circ \varphi = uf$. Moreover, if f is a sum of $p \leq +\infty$ squares, we may assume that g is a sum of $2^{m-1}p + 1$ squares.

Let us delay the proof of the lemma and show before how it is used for the:

Proof of Proposition 7.1. We pick a positive semidefinite analytic function $f : W \to \mathbb{R}$ defined on an open neighborhood W of Z, with $\dim\{f = 0\} \leq 1$, whose germ at Z is f_Z.

We start with 7.1(1). By Lemma 7.2, there are an open neighborhood $\Omega \subset W$ of $\{f = 0\} \cap Z$, and the data g, φ and u such that $g \circ \varphi = uf$. Since g is positive semidefinite on $\mathbb{R}^m$, and we suppose h17 holds for $\mathbb{R}^m$, g is a sum of squares. It follows that $f|_\Omega$ is a sum of squares. Since Ω is a neighborhood of $\{f = 0\} \cap Z$, the conditions of Corollary 1.6 hold for f on $W' = (W \setminus \{f = 0\}) \cup \Omega \supset Z$, and we deduce that f is a sum of squares on W'. Hence f_Z is a sum of squares, and we have proved 7.1(1).

The proof of 7.1(2) is quite the same. Suppose finiteness for $\mathbb{R}^m$, which implies $p = p(\mathbb{R}^m) < +\infty$ (Proposition 1.16). If f is a sum of squares, Lemma 7.2 gives a g which is also a sum of squares, hence a sum of p squares. By Proposition 1.5, g is a sum of $2^{m-1}p$ squares with controlled bad set in a neighborhood of its zero set. Thus, after shrinking Ω, $f|_\Omega$ is a sum of $2^{m-1}p$ squares with controlled bad set. Finally, we can apply Theorem 1.4 to f on $W' = (W \setminus \{f = 0\}) \cup \Omega$, to conclude that f is a sum of $2^{m-1}p + 1$ squares with controlled bad set. $\square$

Thus, we are left with the:

Proof of Lemma 7.2. We will simplify first the data. Shrink W to suppose that all irreducible components of $\{f = 0\}$ meet Z, and pick an analytic function $h : \mathbb{R}^m \to \mathbb{R}$ which is $\geq \frac{1}{2}$ on Z and $\leq -\frac{1}{2}$ on $\mathbb{R}^m \setminus W$. Then, h does not vanish on any irreducible component of $\{f = 0\}$, and $D = \{f = h = 0\}$ is a discrete closed subset of $\mathbb{R}^m$. Clearly, D contains $\mathrm{Cl}_{\mathbb{R}^m}\{f = 0, h > 0\} \setminus \{h > 0\}$, which consequently is also a discrete subset $\{a_i : i \geq 1\}$ of $\mathbb{R}^m$. Now consider a line that does not meet $\mathrm{Cl}_{\mathbb{R}^m}\{f = 0, h > 0\}$, which is possible because the codimension of this set is ≥ 2; after a linear change of coordinates we can suppose the line is the x_m-axis. Next, consider a smooth diffeomorphism of $\mathbb{R}^m$ which is the identity near $Z' = Z \cap \{f = 0\}$, and moves

$$a_1, a_2, a_3, \ldots \mapsto (0, \ldots, 0, 1), (0, \ldots, 0, 2), (0, \ldots, 0, 3), \ldots$$

By analytic approximation [BKS, §3 Prop.3], we get an analytic diffeomorphism of $\mathbb{R}^m$ that keeps Z' off the x_m-axis and also moves each a_i to the point $(0, \ldots, 0, i)$. In other words, we can simply assume $a_i = (0, \ldots, 0, i)$ for all i. Now, we consider the analytic diffeomorphism of $\mathbb{R}^m$ (see the proof of Proposition 6.4):

$$(x', x_m) \mapsto \left(\frac{\sqrt{1 + x_m^2}}{\delta(x_m)}\, x', x_m \right),$$

where $\delta(x_m)$ is an analytic function such that $0 < \delta(x_m) < \mathrm{dist}\big(Z', (0, \ldots, 0, x_m)\big)$. This diffeomorphism moves Z' off $\{\|x'\|^2 < 1 + x_m^2\}$, and fixes the b_i's. Hence, we can assume $Z' \cap \{\|x'\|^2 < 1 + x_m^2\} = \emptyset$. We denote $V = \mathbb{R}^m \setminus \{x_m \geq 0,\ \|x'\|^2 \leq -1 + x_m^2\}$; note that $Z' \subset V$. As the a_i's are not in V,

$$V \cap \{f = 0, h > 0\} = V \cap \mathrm{Cl}_{\mathbb{R}^m}\{f = 0, h > 0\},$$

and $V \cap \{f = 0, h > 0\}$ is closed in V. Finally, V is analytically diffeomorphic to $\mathbb{R}^m$ via:

$$(x', x_m) \mapsto \Big(x', \log\big(\sqrt{1 + \|x'\|^2} - x_m\big)\Big).$$

Take $\Omega = \{h > 0\} \cap V \subset W$.

All this gives an analytic embedding $\varphi : \Omega \to \mathbb{R}^m$ onto an open set $\Omega' \subset \mathbb{R}^m$ and an analytic function $f' = f \circ \varphi^{-1} : \Omega' \to \mathbb{R}$ whose zero set $\{f' = 0\}$ is closed in $\mathbb{R}^m$. Consequently we can define well the following locally principal analytic sheaf:

$$\mathcal{J}_x = \begin{cases} f'\mathcal{O}_{\mathbb{R}^m,x} & \text{if } f'(x) = 0, \\ \mathcal{O}_{\mathbb{R}^m,x} & \text{otherwise} \end{cases}$$

As we have remarked often before, such a sheaf is globally principal, say generated by an analytic function $g : \mathbb{R}^m \to \mathbb{R}$. This function vanishes only on $\{f' = 0\}$, which does not disconnect $\mathbb{R}^m$ (its codimension is ≥ 2), hence g cannot change sign, and we can suppose $g \geq 0$. On the other hand, g and f' generate $\mathcal{J}$ on Ω', hence there is some analytic unit u' on Ω' such that $g|\Omega' = uf'$; since both f' and g are ≥ 0, u' is strictly positive. To conclude, take $u = u' \circ \varphi$.

Now, suppose f is a sum of p squares on W. We deduce that f' is a sum of p squares on Ω', and consequently also $g|_{\Omega'}$. As Ω' is a neighborhood of the zero set of g, Corollary 1.6 says that g is a sum $2^{m-1}p + 1$ of squares on $\mathbb{R}^m$. We are done. □

We end the section and the paper with the proofs of the results for $\mathbb{R}^3$ stated in the introduction:

Proof of Proposition 1.11. Suppose h17 for $\mathbb{R}^3$, and let $Z \subset \mathbb{R}^3$ be closed. We must show that any positive semidefinite analytic germ f_Z is a sum of squares. To that end, let $f : \Omega \to \mathbb{R}$ be a positive semidefinite analytic function whose germ at Z is f_Z. By Lemma 4.1, we have $f = \big(h_1^2 + h_2^2 + h_3^2 + h_4^2\big)f'$, where f' is positive semidefinite and its zero set has codimension ≥ 2 in $\mathbb{R}^3$, hence dimension ≤ 1. By Proposition 7.1(1), the germ of f' at Z is a sum of squares, hence so is the germ of $f = \big(h_1^2 + h_2^2 + h_3^2 + h_4^2\big)f'$. □

Proof of Proposition 1.17. Suppose finiteness for $\mathbb{R}^3$, so that $p = p(\mathbb{R}^3) < +\infty$ (Proposition 1.16), and let $Z \subset \mathbb{R}^3$ be closed. We are to show that any positive semidefinite analytic germ f_Z which is a sum of squares is a sum of $4p+4 < +\infty$ squares. To that end, let $f : \Omega \to \mathbb{R}$ be a positive semidefinite analytic function which is a sum of squares on Ω, and whose germ at Z is f_Z. By Lemma 4.1, we have $f = \big(h_1^2+h_2^2+h_3^2+h_4^2\big)f'$, where the zero set of f' has codimension ≥ 2 in $\mathbb{R}^3$, hence dimension ≤ 1, and multiplying by $h_1^2 + h_2^2 + h_3^2 + h_4^2$, we see that f' is a sum of squares. By Proposition 7.1(2), f'_Z is a sum of $4p + 1$ squares, hence f_Z is a sum of $4p + 4$ squares (recall that the product of two sums of four squares is again a sum of four squares). □

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